

LOCAL ZETA FUNCTIONS FOR RATIONAL FUNCTIONS AND NEWTON POLYHEDRA

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ABSTRACT. In this article, we introduce a notion of non-degeneracy, with respect to certain Newton polyhedra, for rational functions over non-Archimedean local fields of arbitrary characteristic. We study the local zeta functions attached to non-degenerate rational functions, we show the existence of a meromorphic continuation for these zeta functions, as rational functions of q^{-s} , and give explicit formulas. In contrast with the classical local zeta functions, the meromorphic continuation of zeta functions for rational functions have poles with positive and negative real parts.

1. INTRODUCTION

The local zeta functions in the Archimedean setting, i.e. in \mathbb{R} or \mathbb{C} , were introduced in the 50's by I. M. Gel'fand and G. E. Shilov [10]. An important motivation was that the meromorphic continuation for the local zeta functions implies the existence of fundamental solutions for differential operators with constant coefficients. The meromorphic continuation was established, independently, by M. Atiyah [1] and J. Bernstein [3]. On the other hand, by the middle of the 60's, A. Weil studied local zeta functions, in the Archimedean and non-Archimedean settings, in connection with the Poisson-Siegel formula [26]. In the 70's, using Hironaka's resolution of singularities theorem, J.-I. Igusa developed a uniform theory for local zeta functions and oscillatory integrals attached to polynomials with coefficients in a field of characteristic zero [14], [15]. In the p -adic setting, local zeta functions are connected with the number of solutions of polynomial congruences mod p^m and with exponential sums mod p^m . In addition, there are many intriguing conjectures relating the poles of the local zeta functions with topology of complex singularities, see e.g. [7], [15]. More recently, J. Denef and F. Loeser introduced in [9] the motivic zeta functions which constitute a vast generalization of the p -adic local zeta functions.

In [24] W. Veys and W. A. Zúñiga-Galindo extended Igusa's theory to the case of rational functions, or, more generally, meromorphic functions f/g , with coefficients in a local field of characteristic zero. This generalization is far from being straightforward due to the fact that several new geometric phenomena appear. Also, the oscillatory integrals have two different asymptotic expansions: the 'usual' one when the norm of the parameter tends to infinity, and another one when the norm of the parameter tends to zero. The first asymptotic expansion is controlled by the poles (with negative real parts) of all the twisted local zeta functions associated to the meromorphic functions $f/g - c$, for certain special values c . The second expansion

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is controlled by the poles (with positive real parts) of all the twisted local zeta functions associated to f/g . There are several mathematical and physical motivations for studying these new local zeta functions. For instance, S. Gusein-Zade, I. Luengo and A. Melle-Hernández have studied the complex monodromy (and A'Campo zeta functions attached to it) of meromorphic functions, see e.g. [11], [12], [13]. This work drives naturally to ask about the existence of local zeta functions with poles related with the monodromies studied by the mentioned authors. From a physical perspective, the local zeta functions attached to meromorphic functions are very alike to parametric Feynman integrals and to p -adic string amplitudes, see e.g. [2], [4], [5], [20]. For instance in [20, Section 3.15], M. Marcolli pointed out explicitly that the motivic Igusa zeta function constructed by J. Denef and F. Loeser may provide the right tool for a motivic formulation of the dimensionally regularized parametric Feynman integrals.

This article aims to study the local zeta functions attached to a rational function f/g with coefficients in a local field of arbitrary characteristic, when f/g is non-degenerate with respect to a certain Newton polyhedron. In [18] E. León-Cardenal and W. A. Zúñiga-Galindo studied similar matters. In this article, we present a more suitable and general notion of non-degeneracy which allows us to study the local zeta functions attached to much larger class of rational functions. Our article is organized as follows. In Section 2 we summarize some basic aspects about non-Archimedean local fields and compute some π -adic integrals that are needed in the article. In Section 3 we review some basic aspects about polyhedral subdivisions and Newton polyhedra, we also introduce a notion of non-degeneracy for polynomials mappings. It seems that our notion of non-degeneracy is a new one. In Section 4 we study the meromorphic continuation for multivariate local zeta functions attached to non-degenerate mappings. These local zeta functions were introduced by L. Loeser in [19]. We give a very general geometric description of the poles of the meromorphic continuation of these functions, see Theorem 1. Our results extend some of the well-known results due to Hoornaert and Denef [8], and Bories [6]. In Section 5 we study the local zeta functions attached to rational functions satisfying a suitable non-degeneracy condition. In Theorem 2, we give a geometric description of the poles of the meromorphic continuation of these functions. The real parts of the poles of the meromorphic continuation of these functions are positive and negative rational numbers. Finally, in Section 6, we describe the ‘smallest positive and largest negative poles’ appearing in the meromorphic continuation of these new local zeta functions, see Theorems 3, 4.

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2. PRELIMINARIES

In this article we work with a non-discrete locally compact field K of arbitrary characteristic. We will say that a such field is a *non-Archimedean local field* of arbitrary characteristic. By a well-known classification theorem, a non-Archimedean local field is a finite extension of \mathbb{Q}_p , the field of p -adic numbers, or is *the field of formal Laurent series* $\mathbb{F}_q((T))$ over a finite field \mathbb{F}_q . In the first case we say that K is a *p -adic field*. For further details the reader may consult [25, Chapter 1].

Let K be a non-Archimedean local field of arbitrary characteristic and let \mathcal{O}_K be the ring of integers of K and let the residue field of K be \mathbb{F}_q , the finite field with

$q = p^m$ elements, where p is a prime number. For $z \in K \setminus \{0\}$, let $\text{ord}(z) \in \mathbb{Z} \cup \{+\infty\}$ denote the *valuation* of z , let $|z|_K = q^{-\text{ord}(z)}$ denote the normalized *absolute value* (or *norm*), and let $\text{ac}(z) = z\pi^{-\text{ord}(z)}$ denote the *angular component*, where π is a fixed uniformizing parameter of K . We extend the norm $|\cdot|_K$ to K^n by taking $\|(x_1, \dots, x_n)\|_K := \max\{|x_1|_K, \dots, |x_n|_K\}$. Then $(K^n, \|\cdot\|_K)$ is a complete metric space and the metric topology is equal to the product topology.

Along this paper, vectors will be written in boldface, so for instance we will write $\mathbf{b} := (b_1, \dots, b_l)$ where l is a positive integer. For polynomials we will use $\mathbf{x} = (x_1, \dots, x_n)$, thus $h(\mathbf{x}) = h(x_1, \dots, x_n)$. For each n -tuple of natural numbers $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{N}^n$, we will denote by $\sigma(\mathbf{k})$ the sum of all its components i.e. $\sigma(\mathbf{k}) = k_1 + k_2 + \dots + k_n$. Furthermore, we will use the notation $|d\mathbf{x}|_K$ for the Haar measure on $(K^n, +)$ normalized so that the measure of \mathcal{O}_K^n is equal to one. In dimension one, we will use the notation $|dx|_K$.

2.1. Multivariate local zeta functions. We denote by $\mathcal{S}(K^n)$ the \mathbb{C} -vector space consisting of all \mathbb{C} -valued locally constant functions over K^n with compact support. An element of $\mathcal{S}(K^n)$ is called a *Bruhat-Schwartz function* or a *test function*. Along this article we work with a polynomial mapping $\mathbf{h} = (h_1, \dots, h_r) : K^n \rightarrow K^r$ such that each $h_i(\mathbf{x})$ is a non-constant polynomial in $\mathcal{O}_K[x_1, \dots, x_n] \setminus \pi\mathcal{O}_K[x_1, \dots, x_n]$ and $r \leq n$. Let Φ a Bruhat-Schwartz function and let $\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{C}^r$. The local zeta function associated to Φ and \mathbf{h} is defined as

$$Z_\Phi(\mathbf{s}, \mathbf{h}) = \int_{K^n \setminus D_K} \Phi(\mathbf{x}) \prod_{i=1}^r |h_i(\mathbf{x})|_K^{s_i} |d\mathbf{x}|_K$$

for $\text{Re}(s_i) > 0$ for all i , where $D_K := \cup_{i \in \{1, \dots, r\}} \{\mathbf{x} \in K^n; h_i(\mathbf{x}) = 0\}$. Notice that $Z_\Phi(\mathbf{s}, \mathbf{h})$ converges for $\text{Re}(s_i) > 0$ for all $i = 1, \dots, r$. If Φ is the characteristic function of \mathcal{O}_K^n we use the notation $Z(\mathbf{s}, \mathbf{h})$ instead of $Z_\Phi(\mathbf{s}, \mathbf{h})$. In the case of polynomial mappings with coefficients in a local field of characteristic zero (not necessarily non-Archimedean and without the condition $r \leq n$), the theory of local zeta functions of type $Z_\Phi(\mathbf{s}, \mathbf{h})$ was established by F. Loeser in [19].

Denote by $\overline{\mathbf{x}}$ the image of an element of \mathcal{O}_K^n under the canonical homomorphism $\mathcal{O}_K^n \rightarrow \mathcal{O}_K^n / (\pi\mathcal{O}_K)^n \cong \mathbb{F}_q^n$, we call $\overline{\mathbf{x}}$ the *reduction modulo π* of \mathbf{x} . Given $h(\mathbf{x}) \in \mathcal{O}_K[x_1, \dots, x_n]$, we denote by $\overline{h}(\mathbf{x})$ the polynomial obtained by reducing modulo π the coefficients of $h(\mathbf{x})$. Furthermore if $\mathbf{h} = (h_1, \dots, h_r)$ is a polynomial mapping with $h_i \in \mathcal{O}_K[x_1, \dots, x_n]$ for all i , then $\overline{\mathbf{h}} := (\overline{h}_1, \dots, \overline{h}_r)$ denotes the polynomial mapping obtained by reducing modulo π all the components of \mathbf{h} .

2.2. Some π -adic integrals. Let $\mathbf{h} = (h_1, h_2, \dots, h_r)$ be a polynomial mapping as above. For $\mathbf{a} \in (\mathcal{O}_K^\times)^n$, we set

$$(2.1) \quad J_{\mathbf{a}}(\mathbf{s}, \mathbf{h}) := \int_{\mathbf{a} + (\pi\mathcal{O}_K)^n \setminus D_K} \prod_{i=1}^r |h_i(\mathbf{x})|_K^{s_i} |d\mathbf{x}|_K,$$

$\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{C}^r$ with $\text{Re}(s_i) > 0$, $i = 1, \dots, r$.

The Jacobian matrix of \mathbf{h} at \mathbf{a} is $Jac(\mathbf{h}, \mathbf{a}) = \left[\frac{\partial h_i}{\partial x_j}(\mathbf{a}) \right]_{\substack{1 \leq i \leq r \\ 1 \leq j \leq n}}$ with $r \leq n$. In a similar way we define the Jacobian matrix of $\overline{\mathbf{h}}$ at $\overline{\mathbf{a}}$. For every non-empty subset I of $\{1, \dots, r\}$ we set $Jac(\overline{\mathbf{h}}_I, \overline{\mathbf{a}}) := \left[\frac{\partial \overline{h}_i}{\partial x_j}(\overline{\mathbf{a}}) \right]_{\substack{i \in I \\ 1 \leq j \leq n}}$.

Lemma 1. *Let I be the subset of $\{1, \dots, r\}$ such that $\bar{h}_i(\bar{\mathbf{a}}) = 0 \Leftrightarrow i \in I$. Assume that $\mathbf{a} \notin D_K$ and that $\text{Jac}(\bar{\mathbf{h}}_I, \bar{\mathbf{a}})$ has rank $m = \text{Card}(I)$ for $I \neq \emptyset$. Then $J_{\mathbf{a}}(\mathbf{s}, \mathbf{h})$ equals*

$$\begin{cases} q^{-n} & \text{if } I = \emptyset \\ q^{-n} \prod_{i \in I} \frac{(q-1)q^{-1-s_i}}{1-q^{-1-s_i}} & \text{if } I \neq \emptyset. \end{cases}$$

Proof. By change of variables we get

$$J_{\mathbf{a}}(\mathbf{s}, \mathbf{h}) = q^{-n} \int_{\mathcal{O}_K^n \setminus \cup_{i \in \{1, \dots, r\}} \{\mathbf{x} \in K^n, h_i(\pi \mathbf{x} + \mathbf{a}) = 0\}} \prod_{i=1}^r |h_i(\pi \mathbf{x} + \mathbf{a})|_K^{s_i} d\mathbf{x}_K.$$

We first consider the case $I = \emptyset$. Then $h_i(\mathbf{a}) \not\equiv 0 \pmod{\pi}$, thus $|h_i(\pi \mathbf{x} + \mathbf{a})|_K = 1$, and $J_{\mathbf{a}}(\mathbf{s}, \mathbf{h}) = q^{-n}$. In the case $I \neq \emptyset$, by reordering I (if necessary) we can suppose that $I = \{1, \dots, m\}$ with $m \leq r$. Integral $J_{\mathbf{a}}(\mathbf{s}, \mathbf{h})$ is computed by changing variables as $\mathbf{y} = \phi(\mathbf{x})$ with

$$\mathbf{y}_i = \phi_i(\mathbf{x}) := \begin{cases} \frac{h_i(\mathbf{a} + \pi \mathbf{x}) - h_i(\mathbf{a})}{\pi} & \text{if } i = 1, \dots, m \\ x_i & \text{if } i \geq m+1. \end{cases}$$

By using that rank of $\text{Jac}(\bar{\mathbf{h}}_I, \bar{\mathbf{a}})$ is m we get that $\det \left[\frac{\partial \phi_i}{\partial x_j}(\mathbf{0}) \right]_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m}} \not\equiv 0 \pmod{\pi}$, which implies that $\mathbf{y} = \phi(\mathbf{x})$ gives a measure-preserving map from \mathcal{O}_K^n to itself (see e.g. [15, Lemma 7.4.3]), hence

$$J_{\mathbf{a}}(\mathbf{s}, \mathbf{h}) = q^{-n} \prod_{i=1}^m \int_{\mathcal{O}_K \setminus \{\pi y_i + h_i(\mathbf{a}) = 0\}} |\pi y_i + h_i(\mathbf{a})|_K^{s_i} dy_i =: q^{-n} \prod_{i=1}^m J'_{\mathbf{a}}(y_i).$$

To prove the announced formula we compute integrals $J'_{\mathbf{a}}(y_i)$. Now, since $h_i(\mathbf{a}) \equiv 0 \pmod{\pi}$, by taking $z_i = \pi y_i + h_i(\mathbf{a})$ in $J'_{\mathbf{a}}(y_i)$, we obtain

$$J'_{\mathbf{a}}(y_i) = q^{-s_i} \int_{\mathcal{O}_K \setminus \{0\}} |z_i|_K^{s_i} dz_i = \frac{(q-1)q^{-1-s_i}}{1-q^{-1-s_i}}.$$

Therefore

$$(2.2) \quad J_{\mathbf{a}}(\mathbf{s}, \mathbf{h}) = \begin{cases} q^{-n} & I = \emptyset \\ q^{-n} \prod_{i \in I} \frac{(q-1)q^{-1-s_i}}{1-q^{-1-s_i}} & I \neq \emptyset. \end{cases}$$

□

Remark 1. *If in integral (2.1), we replace $h_i(\mathbf{x})$ by $h_i(\mathbf{x}) + \pi g_i(\mathbf{x})$, where each $g_i(\mathbf{x})$ is a polynomial with coefficients in \mathcal{O}_K , then the formulas given in Lemma 1 are valid.*

For every subset $I \subseteq \{1, \dots, r\}$ we set

$$(2.3) \quad \bar{V}_I := \{\bar{\mathbf{z}} \in (\mathbb{F}_q^\times)^n; \bar{h}_i(\bar{\mathbf{z}}) = 0 \Leftrightarrow i \in I\}.$$

To simplify the notation we will denote $\bar{V}_{\{1, \dots, r\}}$ as \bar{V} .

Lemma 2. Let $\mathbf{h} = (h_1, \dots, h_r)$ with $r \leq n$, be as before. Assume that for all $I \neq \emptyset$ if $\overline{V}_I \neq \emptyset$, then

$$\text{rank}_{\mathbb{F}_q} \left[\frac{\partial \overline{h}_i}{\partial x_j}(\overline{\mathbf{a}}) \right]_{i \in I, j \in \{1, \dots, n\}} = \text{Card}(I), \text{ for any } \overline{\mathbf{a}} \in \overline{V}_I.$$

Set

$$L(\mathbf{s}, \mathbf{h}) := \int_{(\mathcal{O}_K^\times)^n \setminus D_K} \prod_{i=1}^r |h_i(\mathbf{x})|_K^{s_i} d\mathbf{x}_K, \quad \mathbf{s} = (s_1, \dots, s_r) \in \mathbb{C}^r,$$

for $\text{Re}(s_i) > 0$ for all i . Then, with the convention that $\prod_{i \in I} \frac{(q-1)q^{-1-s_i}}{1-q^{-1-s_i}} = 1$ when $I = \emptyset$, we have

$$L(\mathbf{s}, \mathbf{h}) = q^{-n} \sum_{I \subseteq \{1, \dots, r\}} \text{Card}(\overline{V}_I) \prod_{i \in I} \frac{(q-1)q^{-1-s_i}}{1-q^{-1-s_i}}.$$

Proof. Note that $L(\mathbf{s}, \mathbf{h})$ can be expressed as a finite sum of integrals

$$J_{\mathbf{a}}(\mathbf{s}, \mathbf{h}) = \int_{\mathbf{a} + (\pi \mathcal{O}_K)^n \setminus D_K} \prod_{i=1}^r |h_i(\mathbf{x})|_K^{s_i} d\mathbf{x}_K,$$

where \mathbf{a} runs through a fixed set of representatives \mathcal{R} in $(\mathcal{O}_K^\times)^n$ of $(\mathbb{F}_q^\times)^n$. Then $L(\mathbf{s}, \mathbf{h})$ is equals

$$\begin{aligned} & \sum_{\overline{\mathbf{a}} \in \overline{V}_\emptyset} \int_{\mathbf{a} + (\pi \mathcal{O}_K)^n \setminus D_K} \prod_{i=1}^r |h_i(\mathbf{x})|_K^{s_i} d\mathbf{x}_K \\ & + \sum_{\substack{I \subseteq \{1, \dots, r\} \\ I \neq \emptyset}} \sum_{\overline{\mathbf{a}} \in \overline{V}_I} \int_{\mathbf{a} + (\pi \mathcal{O}_K)^n \setminus D_K} \prod_{i=1}^r |h_i(\mathbf{x})|_K^{s_i} d\mathbf{x}_K \\ & + \sum_{\overline{\mathbf{a}} \in \overline{V}} \int_{\mathbf{a} + (\pi \mathcal{O}_K)^n \setminus D_K} \prod_{i=1}^r |h_i(\mathbf{x})|_K^{s_i} d\mathbf{x}_K \\ & =: J(\mathbf{s}, \overline{V}_\emptyset) + \sum_{\substack{I \subseteq \{1, \dots, r\} \\ I \neq \emptyset}} J(\mathbf{s}, \overline{V}_I) + J(\mathbf{s}, \overline{V}), \end{aligned}$$

with the convention that if $\overline{V}_I = \emptyset$, then $\sum_{\overline{\mathbf{a}} \in \overline{V}_I} \int_{\mathbf{a} + (\pi \mathcal{O}_K)^n \setminus D_K} \cdot = 0$. Notice that

$$(2.4) \quad J(\mathbf{s}, \overline{V}_\emptyset) = q^{-n} \text{Card}(\overline{V}_\emptyset).$$

Thus we may assume that $I \neq \emptyset$. In the calculation of $J(\mathbf{s}, \overline{V}_I)$ we use the following result:

Claim.

$$\sum_{\overline{\mathbf{a}} \in \overline{V}_I} \int_{\mathbf{a} + (\pi \mathcal{O}_K)^n \setminus D_K} \prod_{i=1}^r |h_i(\mathbf{x})|_K^{s_i} d\mathbf{x}_K = \sum_{\substack{\overline{\mathbf{a}} \in \overline{V}_I \\ \mathbf{a} \notin D_K}} \int_{\mathbf{a} + (\pi \mathcal{O}_K)^n \setminus D_K} \prod_{i=1}^r |h_i(\mathbf{x})|_K^{s_i} d\mathbf{x}_K.$$

The Claim follows from the following reasoning. The analytic mapping $h_1 \cdots h_r : \mathbf{a} + (\pi \mathcal{O}_K)^n \rightarrow K$ is not identically zero, otherwise by [15, Lemma 2.1.3], the polynomial $(h_1 \cdots h_r)(\mathbf{x})$ would be the constant polynomial zero contradicting the

hypothesis that all the h_i 's are non-constant polynomials. Hence there exists an element $\mathbf{b} \in \mathbf{a} + (\pi\mathcal{O}_K)^n$ such that $(h_1 \cdots h_r)(\mathbf{b}) \neq 0$. Finally, we use the fact that every point of a ball is its center, which implies that $\mathbf{a} + (\pi\mathcal{O}_K)^n = \mathbf{b} + (\pi\mathcal{O}_K)^n$.

By using Lemma 1,

$$(2.5) \quad J(\mathbf{s}, \overline{V}_I) = q^{-n} \text{Card}(\overline{V}_I) \prod_{i \in I} \frac{(q-1)q^{-1-s_i}}{1 - q^{-1-s_i}}.$$

The formula for $J(\mathbf{s}, \overline{V})$ is a special case of formula (2.5):

$$(2.6) \quad J(\mathbf{s}, \overline{V}) = q^{-n} \text{Card}(\overline{V}) \prod_{i \in \{1, \dots, r\}} \frac{(q-1)q^{-1-s_i}}{1 - q^{-1-s_i}}.$$

□

Remark 2. In integral $L(\mathbf{s}, \mathbf{h})$ we can replace \mathbf{h} by $\mathbf{h} + \pi\mathbf{g}$, where \mathbf{g} is a polynomial mapping over \mathcal{O}_K , and the formulas given in Lemma 2 remain valid.

3. POLYHEDRAL SUBDIVISIONS OF \mathbb{R}_+^n AND NON-DEGENERACY CONDITIONS

In this section we review, without proofs, some well-known results about Newton polyhedra and non-degeneracy conditions that we will use along the article. Our presentation follows closely [27], [21].

3.1. Newton polyhedra. We set $\mathbb{R}_+ := \{x \in \mathbb{R}; x \geq 0\}$. Let G be a non-empty subset of \mathbb{N}^n . The *Newton polyhedron* $\Gamma = \Gamma(G)$ associated to G is the convex hull in \mathbb{R}_+^n of the set $\cup_{\mathbf{m} \in G} (\mathbf{m} + \mathbb{R}_+^n)$. For instance classically one associates a *Newton polyhedron* $\Gamma(h)$ (at the origin) to a polynomial function $h(\mathbf{x}) = \sum_{\mathbf{m}} c_{\mathbf{m}} \mathbf{x}^{\mathbf{m}}$ ($\mathbf{x} = (x_1, \dots, x_n)$, $h(\mathbf{0}) = 0$), where $G = \text{supp}(h) := \{\mathbf{m} \in \mathbb{N}^n; c_{\mathbf{m}} \neq 0\}$. Further we will associate more generally a Newton polyhedron to a polynomial mapping.

We fix a Newton polyhedron Γ as above. We first collect some notions and results about Newton polyhedra that will be used in the next sections. Let $\langle \cdot, \cdot \rangle$ denote the usual inner product of \mathbb{R}^n , and identify the dual space of \mathbb{R}^n with \mathbb{R}^n itself by means of it.

Let H be the hyperplane $H = \{\mathbf{x} \in \mathbb{R}^n; \langle \mathbf{x}, \mathbf{b} \rangle = c\}$, H determines two closed half-spaces

$$H^+ = \{\mathbf{x} \in \mathbb{R}^n; \langle \mathbf{x}, \mathbf{b} \rangle \geq c\} \text{ and } H^- = \{\mathbf{x} \in \mathbb{R}^n; \langle \mathbf{x}, \mathbf{b} \rangle \leq c\}.$$

We say that H is a *supporting hyperplane* of $\Gamma(h)$ if $\Gamma(h) \cap H \neq \emptyset$ and $\Gamma(h)$ is contained in one of the two closed half-spaces determined by H . By a *proper face* τ of $\Gamma(h)$, we mean a non-empty convex set τ obtained by intersecting $\Gamma(h)$ with one of its supporting hyperplanes. By the *faces* of $\Gamma(h)$ we will mean the proper faces of $\Gamma(h)$ and the whole the polyhedron $\Gamma(h)$. By *dimension of a face* τ of $\Gamma(h)$ we mean the dimension of the affine hull of τ , and its *codimension* is $\text{cod}(\tau) = n - \dim(\tau)$, where $\dim(\tau)$ denotes the dimension of τ . A face of codimension one is called a *facet*.

For $\mathbf{a} \in \mathbb{R}_+^n$, we define

$$d(\mathbf{a}, \Gamma) = \min_{\mathbf{x} \in \Gamma} \langle \mathbf{a}, \mathbf{x} \rangle,$$

and the *first meet locus* $F(\mathbf{a}, \Gamma)$ of \mathbf{a} as

$$F(\mathbf{a}, \Gamma) := \{\mathbf{x} \in \Gamma; \langle \mathbf{a}, \mathbf{x} \rangle = d(\mathbf{a}, \Gamma)\}.$$

The first meet locus is a face of Γ . Moreover, if $\mathbf{a} \neq \mathbf{0}$, $F(\mathbf{a}, \Gamma)$ is a proper face of Γ .

If $\Gamma = \Gamma(h)$, we define the *face function* $h_{\mathbf{a}}(\mathbf{x})$ of $h(\mathbf{x})$ with respect to \mathbf{a} as

$$h_{\mathbf{a}}(\mathbf{x}) = h_{F(\mathbf{a}, \Gamma)}(\mathbf{x}) = \sum_{\mathbf{m} \in F(\mathbf{a}, \Gamma)} c_{\mathbf{m}} \mathbf{x}^{\mathbf{m}}.$$

In the case of functions having subindices, say $h_i(\mathbf{x})$, we will use the notation $h_{i, \mathbf{a}}(\mathbf{x})$ for the face function of $h_i(\mathbf{x})$ with respect to \mathbf{a} . Notice that $h_{\mathbf{0}}(\mathbf{x}) = h_{F(\mathbf{0}, \Gamma)}(\mathbf{x}) = \sum_{\mathbf{m}} c_{\mathbf{m}} \mathbf{x}^{\mathbf{m}}$.

3.2. Polyhedral Subdivisions Subordinate to a Polyhedron. We define an equivalence relation in \mathbb{R}_+^n by taking $\mathbf{a} \sim \mathbf{a}' \Leftrightarrow F(\mathbf{a}, \Gamma) = F(\mathbf{a}', \Gamma)$. The equivalence classes of \sim are sets of the form

$$\Delta_{\tau} = \{\mathbf{a} \in \mathbb{R}_+^n; F(\mathbf{a}, \Gamma) = \tau\},$$

where τ is a face of Γ .

We recall that the *cone strictly spanned* by the vectors $\mathbf{a}_1, \dots, \mathbf{a}_l \in \mathbb{R}_+^n \setminus \{0\}$ is the set $\Delta = \{\lambda_1 \mathbf{a}_1 + \dots + \lambda_l \mathbf{a}_l; \lambda_i \in \mathbb{R}_+, \lambda_i > 0\}$. If $\mathbf{a}_1, \dots, \mathbf{a}_l$ are linearly independent over \mathbb{R} , Δ is called a *simplicial cone*. If $\mathbf{a}_1, \dots, \mathbf{a}_l \in \mathbb{Z}^n$, we say Δ is a *rational cone*. If $\{\mathbf{a}_1, \dots, \mathbf{a}_l\}$ is a subset of a basis of the \mathbb{Z} -module \mathbb{Z}^n , we call Δ a *simple cone*.

A precise description of the geometry of the equivalence classes modulo \sim is as follows. Each *facet* γ of Γ has a unique vector $\mathbf{a}(\gamma) = (a_{\gamma,1}, \dots, a_{\gamma,n}) \in \mathbb{N}^n \setminus \{0\}$, whose nonzero coordinates are relatively prime, which is perpendicular to γ . We denote by $\mathfrak{D}(\Gamma)$ the set of such vectors. The equivalence classes are rational cones of the form

$$\Delta_{\tau} = \left\{ \sum_{i=1}^r \lambda_i \mathbf{a}(\gamma_i); \lambda_i \in \mathbb{R}_+, \lambda_i > 0 \right\},$$

where τ runs through the set of faces of Γ , and γ_i , $i = 1, \dots, r$ are the facets containing τ . We note that $\Delta_{\tau} = \{0\}$ if and only if $\tau = \Gamma$. The family $\{\Delta_{\tau}\}_{\tau}$, with τ running over the proper faces of Γ , is a partition of $\mathbb{R}_+^n \setminus \{0\}$; we call this partition a *polyhedral subdivision of \mathbb{R}_+^n subordinate to Γ* . We call $\{\overline{\Delta}_{\tau}\}_{\tau}$, the family formed by the topological closures of the Δ_{τ} , a *fan subordinate to Γ* .

Each cone Δ_{τ} can be partitioned into a finite number of simplicial cones $\Delta_{\tau,i}$. In addition, the subdivision can be chosen such that each $\Delta_{\tau,i}$ is spanned by part of $\mathfrak{D}(\Gamma)$. Thus from the above considerations we have the following partition of $\mathbb{R}_+^n \setminus \{0\}$:

$$\mathbb{R}_+^n \setminus \{0\} = \bigcup_{\tau} \left(\bigcup_{i=1}^{l_{\tau}} \Delta_{\tau,i} \right),$$

where τ runs over the proper faces of Γ , and each $\Delta_{\tau,i}$ is a simplicial cone contained in Δ_{τ} . We will say that $\{\Delta_{\tau,i}\}$ is a *simplicial polyhedral subdivision of \mathbb{R}_+^n subordinate to Γ* , and that $\{\overline{\Delta}_{\tau,i}\}$ is a *simplicial fan subordinate to Γ* .

By adding new rays, each simplicial cone can be partitioned further into a finite number of simple cones. In this way we obtain a *simple polyhedral subdivision of \mathbb{R}_+^n subordinate to Γ* , and a *simple fan subordinate to Γ* (or a *complete regular fan*) (see e.g. [16]).

3.3. The Newton polyhedron associated to a polynomial mapping. Let $\mathbf{h} = (h_1, \dots, h_r)$, $\mathbf{h}(\mathbf{0}) = 0$, be a non-constant polynomial mapping. In this article we associate to \mathbf{h} a Newton polyhedron $\Gamma(\mathbf{h}) := \Gamma(\prod_{i=1}^r h_i(\mathbf{x}))$. From a geometrical point of view, $\Gamma(\mathbf{h})$ is the Minkowski sum of the $\Gamma(h_i)$, for $i = 1, \dots, r$, (see e.g. [21], [22]). By using the results previously presented, we can associate to $\Gamma(\mathbf{h})$ a simplicial polyhedral subdivision $\mathcal{F}(\mathbf{h})$ of \mathbb{R}_+^n subordinate to $\Gamma(\mathbf{h})$.

Remark 3. A basic fact about the Minkowski sum operation is the additivity of the faces. From this fact follows:

- (1) $F(\mathbf{a}, \Gamma(\mathbf{h})) = \sum_{j=1}^r F(\mathbf{a}, \Gamma(h_j))$, for $\mathbf{a} \in \mathbb{R}_+^n$;
- (2) $d(\mathbf{a}, \Gamma(\mathbf{h})) = \sum_{j=1}^r d(\mathbf{a}, \Gamma(h_j))$, for $\mathbf{a} \in \mathbb{R}_+^n$;
- (3) let τ be a proper face of $\Gamma(\mathbf{h})$, and let τ_j be proper face of $\Gamma(h_j)$, for $i = 1, \dots, r$. If $\tau = \sum_{j=1}^r \tau_j$, then $\Delta_\tau \subseteq \overline{\Delta}_{\tau_j}$, for $i = 1, \dots, r$.

Remark 4. Note that the equivalence relation,

$$\mathbf{a} \sim \mathbf{a}' \Leftrightarrow F(\mathbf{a}, \Gamma(\mathbf{h})) = F(\mathbf{a}', \Gamma(\mathbf{h})),$$

used in the construction of a polyhedral subdivision of \mathbb{R}_+^n subordinate to $\Gamma(\mathbf{h})$ can be equivalently defined in the following form:

$$\mathbf{a} \sim \mathbf{a}' \Leftrightarrow F(\mathbf{a}, \Gamma(h_j)) = F(\mathbf{a}', \Gamma(h_j)), \text{ for each } j = 1, \dots, r.$$

This last definition is used in Oka's book [21].

3.4. Non-degeneracy Conditions. For $K = \mathbb{Q}_p$, Denef and Hoornaert in [8, Theorem 4.2] gave an explicit formula for $Z(\mathbf{s}, \mathbf{h})$, in the case $r = 1$ with \mathbf{h} a non-degenerate polynomial with respect to its Newton polyhedron $\Gamma(\mathbf{h})$. This explicit formula can be generalized to the case $r \geq 1$ by using the condition of non-degeneracy for polynomial mappings introduced here.

Definition 1. Let $\mathbf{h} = (h_1, \dots, h_r)$, $\mathbf{h}(\mathbf{0}) = 0$, be a polynomial mapping with $r \leq n$ as in Subsection 2.1 and let $\Gamma(\mathbf{h})$ be the Newton polyhedron of \mathbf{h} at the origin. The mapping \mathbf{h} is called non-degenerate over \mathbb{F}_q with respect to $\Gamma(\mathbf{h})$, if for every vector $\mathbf{k} \in \mathbb{R}_+^n$ and for any non-empty subset $I \subseteq \{1, \dots, r\}$, it verifies that

$$(3.1) \quad \text{rank}_{\mathbb{F}_q} \left[\frac{\partial \bar{h}_{i,\mathbf{k}}}{\partial x_j}(\bar{\mathbf{z}}) \right]_{i \in I, j \in \{1, \dots, n\}} = \text{Card}(I)$$

for any

$$(3.2) \quad \bar{\mathbf{z}} \in \{\bar{\mathbf{z}} \in (\mathbb{F}_q^\times)^n; \bar{h}_{i,\mathbf{k}}(\bar{\mathbf{z}}) = 0 \Leftrightarrow i \in I\}.$$

We notice that above notion is different to the those introduced in [23], [27]. The notion introduced here is similar to the usual notion given by Khovansky, see [17], [21]. For a discussion about the relation between Khovansky's non-degeneracy notion and other similar notions we refer the reader to [23].

Let Δ be a rational simplicial cone spanned by \mathbf{w}_i , $i = 1, \dots, e_\Delta$. We define the barycenter of Δ as $b(\Delta) = \sum_{i=1}^{e_\Delta} \mathbf{w}_i$. Set $b(\{\mathbf{0}\}) := \mathbf{0}$.

Remark 5. (i) Let $\mathcal{F}(\mathbf{h})$ be a simplicial polyhedral subdivision of \mathbb{R}_+^n subordinate to $\Gamma(\mathbf{h})$. Then, it is sufficient to verify the condition given in Definition 1 for $\mathbf{k} = b(\Delta)$ with $\Delta \in \mathcal{F}(\mathbf{h}) \cup \{\mathbf{0}\}$.

(ii) Notice that our notion of non-degeneracy agrees, in the case $K = \mathbb{Q}_p$, $r = 1$, with the corresponding notion in [8].

Example 1. Set $\mathbf{h} = (h_1, h_2)$ with $h_1(x, y) = x^2 - y$, $h_2(x, y) = x^2 y$ polynomials in $\mathcal{O}_K[x, y]$. Then a simplicial polyhedral subdivision subordinate to $\Gamma(\mathbf{h})$ is given by

Cone	$h_{1,b(\Delta)}$	$h_{2,b(\Delta)}$
$\Delta_1 := (1, 0)\mathbb{R}_{>0}$	y	$x^2 y$
$\Delta_2 := (1, 0)\mathbb{R}_{>0} + (1, 2)\mathbb{R}_{>0}$	y	$x^2 y$
$\Delta_3 := (1, 2)\mathbb{R}_{>0}$	$x^2 - y$	$x^2 y$
$\Delta_4 := (1, 2)\mathbb{R}_{>0} + (0, 1)\mathbb{R}_{>0}$	x^2	$x^2 y$
$\Delta_5 := (0, 1)\mathbb{R}_{>0}$	x^2	$x^2 y$

where $\mathbb{R}_{>0} := \mathbb{R}_+ \setminus \{0\}$. Notice that for every $\mathbf{k} \in \mathbb{R}_+^n \setminus (\{0\} \cup \Delta_3)$ and every non-empty subset $I \subseteq \{1, 2\}$, the subset defined in (3.2) is empty, thus (3.1) is always satisfied. In the case $\mathbf{k} = 0$ and $\mathbf{k} \in \Delta_3$, $h_{1,\mathbf{k}} = x^2 - y$, $h_{2,\mathbf{k}} = x^2 y$, the conditions (3.2)-(3.1) are also verified. Hence \mathbf{h} is non-degenerate over \mathbb{F}_q with respect to $\Gamma(\mathbf{h})$.

Example 2. Let $\mathbf{h} = (h_1(\mathbf{x}), \dots, h_r(\mathbf{x}))$ be a monomial mapping. In this case, $\Gamma(\mathbf{h}) = \mathbf{m}_0 + \mathbb{R}_+^n$ for some nonzero vector \mathbf{m}_0 in \mathbb{N}^n . Then for every vector $\mathbf{k} \in \mathbb{R}_+^n$ $h_{i,\mathbf{k}}(\mathbf{x}) = h_i(\mathbf{x})$ for $i = 1, \dots, r$, and thus the subset in (3.2) is always empty, which implies that condition (3.1) is always satisfied. Therefore any monomial mapping (with $r \leq n$) is non-degenerate over \mathbb{F}_q with respect to its Newton polyhedron.

Example 3. $f(\mathbf{x}), g(\mathbf{x}) \in \mathcal{O}_K[x_1, \dots, x_n] \setminus \pi \mathcal{O}_K[x_1, \dots, x_n]$ such that $g(\mathbf{x}) = \mathbf{x}^{\mathbf{m}_0}$, with $\mathbf{m}_0 \neq 0$, is a monomial. Suppose that f is non-degenerate with respect to $\Gamma(f)$ over \mathbb{F}_q . In this case, $\Gamma((f, g)) = \mathbf{m}_0 + \Gamma(f)$. Then the subset in (3.2) can take three different forms:

$$(i) \{ \bar{\mathbf{z}} \in (\mathbb{F}_q^\times)^n; \bar{f}_{\mathbf{k}}(\bar{\mathbf{z}}) = \bar{g}(\bar{\mathbf{z}}) = 0 \} = \emptyset, (ii) \{ \bar{\mathbf{z}} \in (\mathbb{F}_q^\times)^n; \bar{f}_{\mathbf{k}}(\bar{\mathbf{z}}) = 0 \}, \\ (iii) \{ \bar{\mathbf{z}} \in (\mathbb{F}_q^\times)^n; \bar{g}(\bar{\mathbf{z}}) = 0, \bar{f}_{\mathbf{k}}(\bar{\mathbf{z}}) \neq 0 \} = \emptyset.$$

In the second case, conditions (3.2)-(3.1) are verified due to the hypothesis that f is non-degenerate with respect $\Gamma(f)$ over \mathbb{F}_q . Hence, (f, g) is a non-degenerate mapping over \mathbb{F}_q with respect to $\Gamma((f, g))$ over \mathbb{F}_q .

4. MEROMORPHIC CONTINUATION OF MULTIVARIATE LOCAL ZETA FUNCTIONS

Along this section, we work with a fix simplicial polyhedral subdivision $\mathcal{F}(\mathbf{h})$ subordinate to $\Gamma(\mathbf{h})$. Let $\Delta \in \mathcal{F}(\mathbf{h}) \cup \{0\}$ and $I \subseteq \{1, \dots, r\}$, we put

$$\bar{V}_{\Delta, I} := \{ \bar{\mathbf{z}} \in (\mathbb{F}_q^\times)^n; \bar{h}_{i,b(\Delta)}(\bar{\mathbf{z}}) = 0 \Leftrightarrow i \in I \}.$$

We use the convention $\bar{V}_{\Delta, \{1, \dots, r\}} = \bar{V}_\Delta$. If $\Delta = 0$, then

$$\bar{V}_{0, I} = \{ \bar{\mathbf{z}} \in (\mathbb{F}_q^\times)^n; \bar{h}_i(\bar{\mathbf{z}}) = 0 \Leftrightarrow i \in I \} = \bar{V}_I,$$

where \bar{V}_I is the set defined in (2.3). In particular, $\bar{V}_{0, \{1, \dots, r\}} = \bar{V}$ and

$$\bar{V}_{0, \emptyset} = \{ \bar{\mathbf{z}} \in (\mathbb{F}_q^\times)^n; \bar{h}_i(\bar{\mathbf{z}}) \neq 0, i = 1, \dots, r \} = \bar{V}_\emptyset.$$

If $\mathbf{h} = (h_1, \dots, h_r)$ is non-degenerated polynomial mapping over \mathbb{F}_q with respect to $\Gamma(\mathbf{h})$, then Lemma 2 is true for $\mathbf{h}_{b(\Delta)} = (h_{1,b(\Delta)}, \dots, h_{r,b(\Delta)})$.

Theorem 1. Assume that $\mathbf{h} = (h_1, \dots, h_r)$ is non-degenerated polynomial mapping over \mathbb{F}_q with respect to $\Gamma(\mathbf{h})$, with $r \leq n$ as before. Fix a simplicial polyhedral subdivision $\mathcal{F}(\mathbf{h})$ subordinate to $\Gamma(\mathbf{h})$. Then $Z(\mathbf{s}, \mathbf{h})$ has a meromorphic continuation

to \mathbb{C}^r as a rational function in the variables q^{-s_i} , $i = 1, \dots, r$. In addition, the following explicit formula holds:

$$Z(\mathbf{s}, \mathbf{h}) = L_{\{\mathbf{0}\}}(\mathbf{s}, \mathbf{h}) + \sum_{\Delta \in \mathcal{F}(\mathbf{h})} L_{\Delta}(\mathbf{s}, \mathbf{h}) S_{\Delta},$$

where

$$L_{\{\mathbf{0}\}} = q^{-n} \sum_{I \subseteq \{1, \dots, r\}} \text{Card}(\overline{V}_I) \prod_{i \in I} \frac{(q-1)q^{-1-s_i}}{1-q^{-1-s_i}},$$

$$L_{\Delta} = q^{-n} \sum_{I \subseteq \{1, \dots, r\}} \text{Card}(\overline{V}_{\Delta, I}) \prod_{i \in I} \frac{(q-1)q^{-1-s_i}}{1-q^{-1-s_i}},$$

with the convention that for $I = \emptyset$, $\prod_{i \in I} \frac{(q-1)q^{-1-s_i}}{1-q^{-1-s_i}} := 1$, and

$$S_{\Delta} = \sum_{\mathbf{k} \in \mathbb{N}^n \cap \Delta} q^{-\sigma(\mathbf{k}) - \sum_{i=1}^r d(\mathbf{k}, \Gamma(h_i)) s_i}.$$

Let Δ be the cone strictly positively generated by linearly independent vectors $\mathbf{w}_1, \dots, \mathbf{w}_l \in \mathbb{N}^n \setminus \{\mathbf{0}\}$, then

$$S_{\Delta} = \frac{\sum_{\mathbf{t}} q^{-\sigma(\mathbf{t}) - \sum_{i=1}^r d(\mathbf{t}, \Gamma(h_i)) s_i}}{(1 - q^{-\sigma(\mathbf{w}_1) - \sum_{i=1}^r d(\mathbf{w}_1, \Gamma(h_i)) s_i}) \dots (1 - q^{-\sigma(\mathbf{w}_l) - \sum_{i=1}^r d(\mathbf{w}_l, \Gamma(h_i)) s_i})},$$

where \mathbf{t} runs through the elements of the set

$$(4.1) \quad \mathbb{Z}^n \cap \left\{ \sum_{i=1}^l \lambda_i \mathbf{w}_i; \ 0 < \lambda_i \leq 1 \text{ for } i = 1, \dots, l \right\}.$$

Proof. By using the simplicial polyhedral subdivision $\mathcal{F}(\mathbf{h})$, we have

$$\mathbb{R}_+^n = \{\mathbf{0}\} \sqcup \bigsqcup_{\Delta \in \mathcal{F}(\mathbf{h})} \Delta.$$

We set for $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{N}^n$,

$$E_{\mathbf{k}} := \{(x_1, \dots, x_n) \in \mathcal{O}_K^n; \text{ord}(x_i) = k_i, i = 1, \dots, n\}.$$

Hence

$$Z(\mathbf{s}, \mathbf{h}) = \int_{(\mathcal{O}_K^\times)^n \setminus D_K} \prod_{i=1}^r |h_i(x)|_K^{s_i} |d\mathbf{x}|_K + \sum_{\Delta \in \mathcal{F}(\mathbf{h})} \sum_{\mathbf{k} \in \mathbb{N}^n \cap \Delta} \int_{E_{\mathbf{k}} \setminus D_K} \prod_{i=1}^r |h_i(x)|_K^{s_i} |d\mathbf{x}|_K.$$

For $\Delta \in \mathcal{F}(\mathbf{h})$, $\mathbf{k} \in \mathbb{N}^n \cap \Delta$, and $\mathbf{x} = (x_1, \dots, x_n) \in E_{\mathbf{k}}$, we put $x_j = \pi^{k_j} u_j$ with $u_j \in \mathcal{O}_K^\times$. Then

$$|d\mathbf{x}|_K = q^{-\sigma(\mathbf{k})} |d\mathbf{u}|_K \text{ and } \mathbf{x}^{\mathbf{m}} = x_1^{m_1} \dots x_n^{m_n} = \pi^{\langle \mathbf{k}, \mathbf{m} \rangle} u_1^{m_1} \dots u_n^{m_n}.$$

Fix $i \in \{1, \dots, r\}$ and \mathbf{k} . For $\mathbf{m} \in \text{supp}(h_i)$, the scalar product $\langle \mathbf{k}, \mathbf{m} \rangle$ attains its minimum $d(\mathbf{k}, \Gamma(h_i))$ exactly when $\mathbf{m} \in F(\mathbf{k}, \Gamma(h_i))$, and thus $\langle \mathbf{k}, \mathbf{m} \rangle \geq d(\mathbf{k}, \Gamma(h_i)) + 1$ for $\mathbf{m} \in \text{supp}(h_i) \setminus \text{supp}(h_i) \cap F(\mathbf{k}, \Gamma(h_i))$. This fact implies that

$$\begin{aligned} h_i(\mathbf{x}) &= \pi^{d(\mathbf{k}, \Gamma(h_i))} (h_{i, \mathbf{k}}(\mathbf{u}) + \pi \tilde{h}_{i, \mathbf{k}}(\mathbf{u})) \\ &= \pi^{d(\mathbf{k}, \Gamma(h_i))} (h_{i, b(\Delta)}(\mathbf{u}) + \pi \tilde{h}_{i, \mathbf{k}}(\mathbf{u})), \end{aligned}$$

where $\tilde{h}_{i,\mathbf{k}}(\mathbf{u})$ is a polynomial over \mathcal{O}_K in the variables u_1, \dots, u_n . Note that $h_{i,\mathbf{k}}(\mathbf{u})$ does not depend on $\mathbf{k} \in \Delta$, for this reason we take $h_{i,\mathbf{k}}(\mathbf{u}) = h_{i,b(\Delta)}(\mathbf{u})$. Therefore

$$Z(\mathbf{s}, \mathbf{h}) = L_{\{0\}}(\mathbf{s}, \mathbf{h}) + \sum_{\Delta \in \mathcal{F}(\mathbf{h})} L_{\Delta}(\mathbf{s}, \mathbf{h}) \sum_{\mathbf{k} \in \mathbb{N}^n \cap \Delta} q^{-\sigma(\mathbf{k}) - \sum_{i=1}^r d(\mathbf{k}, \Gamma(h_i))s_i}$$

where

$$L_{\{0\}}(\mathbf{s}, \mathbf{h}) := \int_{(\mathcal{O}_K^\times)^n \setminus D_K} \prod_{i=1}^r |h_i(x)|_K^{s_i} |dx|_K,$$

$$L_{\Delta}(\mathbf{s}, \mathbf{h}) := \int_{(\mathcal{O}_K^\times)^n \setminus D_{\Delta}} \prod_{i=1}^r |h_{i,b(\Delta)}(\mathbf{u}) + \pi \tilde{h}_{i,\mathbf{k}}(\mathbf{u})|_K^{s_i} |d\mathbf{u}|_K$$

with $D_{\Delta} = \bigcup_{i=1}^r \left\{ \mathbf{x} \in (\mathcal{O}_K^\times)^n; h_{i,b(\Delta)}(\mathbf{u}) + \pi \tilde{h}_{i,\mathbf{k}}(\mathbf{u}) = 0 \right\}$. By using the non-degeneracy condition, integrals $L_{\{0\}}(\mathbf{s}, \mathbf{h})$, $L_{\Delta}(\mathbf{s}, \mathbf{h})$ can be computed using Lemma 2 and Remarks 1, 2.

Let Δ be the cone strictly positively generated by linearly independent vectors $\mathbf{w}_1, \dots, \mathbf{w}_l \in \mathbb{N}^n \setminus \{0\}$. If Δ is a simple cone then $\mathbb{N}^n \cap \Delta = (\mathbb{N} \setminus \{0\}) \mathbf{w}_1 + \dots + (\mathbb{N} \setminus \{0\}) \mathbf{w}_l$. By using that the functions $d(\cdot, \Gamma(h_i))$ are linear over each cone Δ , and that

$$\sigma(\mathbf{w}_m) + \sum_{i=1}^r d(\mathbf{w}_m, \Gamma(h_i)) \operatorname{Re}(s_i) > 0, m = 1, \dots, l,$$

since $\operatorname{Re}(s_1), \dots, \operatorname{Re}(s_r) > 0$, we obtain

$$\begin{aligned} S_{\Delta} &= \sum_{\lambda_1, \dots, \lambda_l \in \mathbb{N} \setminus \{0\}} q^{-\sigma(\lambda_1 \mathbf{w}_1 + \dots + \lambda_l \mathbf{w}_l) - \sum_{i=1}^r d(\lambda_1 \mathbf{w}_1 + \dots + \lambda_l \mathbf{w}_l, \Gamma(h_i))s_i} \\ &= \sum_{\lambda_1=1}^{\infty} (q^{-\sigma(\mathbf{w}_1) - \sum_{i=1}^r d(\mathbf{w}_1, \Gamma(h_i))s_i})^{\lambda_1} \dots \sum_{\lambda_l=1}^{\infty} (q^{-\sigma(\mathbf{w}_l) - \sum_{i=1}^r d(\mathbf{w}_l, \Gamma(h_i))s_i})^{\lambda_l} \\ S_{\Delta} &= \frac{q^{-\sigma(\mathbf{w}_1) - \sum_{i=1}^r d(\mathbf{w}_1, \Gamma(h_i))s_i}}{1 - q^{-\sigma(\mathbf{w}_1) - \sum_{i=1}^r d(\mathbf{w}_1, \Gamma(h_i))s_i}} \dots \frac{q^{-\sigma(\mathbf{w}_l) - \sum_{i=1}^r d(\mathbf{w}_l, \Gamma(h_i))s_i}}{1 - q^{-\sigma(\mathbf{w}_l) - \sum_{i=1}^r d(\mathbf{w}_l, \Gamma(h_i))s_i}} \\ &= \frac{\sum_{\mathbf{t}} q^{-\sigma(\mathbf{t}) - \sum_{i=1}^r d(\mathbf{t}, \Gamma(h_i))s_i}}{(1 - q^{-\sigma(\mathbf{w}_1) - \sum_{i=1}^r d(\mathbf{w}_1, \Gamma(h_i))s_i}) \dots (1 - q^{-\sigma(\mathbf{w}_l) - \sum_{i=1}^r d(\mathbf{w}_l, \Gamma(h_i))s_i})}, \end{aligned}$$

where \mathbf{t} runs through the elements of the set (4.1), which consists exactly of one element: $\mathbf{t} = \sum_{i=1}^l \mathbf{w}_i$. We now consider the case in which Δ is a simplicial cone. Note that $\mathbb{N}^n \cap \Delta$ is the disjoint union of the sets

$$\mathbf{t} + \mathbb{N}\mathbf{w}_1 + \dots + \mathbb{N}\mathbf{w}_l,$$

where \mathbf{t} runs through the elements of the set

$$\mathbb{Z}^n \cap \left\{ \sum_{i=1}^l \lambda_i \mathbf{w}_i; 0 < \lambda_i \leq 1 \text{ for } i = 1, \dots, l \right\}.$$

Hence S_{Δ} equals

$$\sum_{\mathbf{t}} q^{-\sigma(\mathbf{t}) - \sum_{i=1}^r d(\mathbf{t}, \Gamma(h_i))s_i} \sum_{\lambda_1, \dots, \lambda_l \in \mathbb{N}} q^{-\sigma(\sum_{j=1}^l \lambda_j \mathbf{w}_j) - \sum_{i=1}^r d(\lambda_1 \mathbf{w}_1 + \dots + \lambda_l \mathbf{w}_l, \Gamma(h_i))s_i},$$

and since $\operatorname{Re}(s_1), \dots, \operatorname{Re}(s_r) > 0$,

$$S_\Delta = \frac{\sum_{\mathbf{t}} q^{-\sigma(\mathbf{t}) - \sum_{i=1}^r d(\mathbf{t}, \Gamma(h_i)) s_i}}{(1 - q^{-\sigma(\mathbf{w}_1) - \sum_{i=1}^r d(\mathbf{w}_1, \Gamma(h_i)) s_i}) \dots (1 - q^{-\sigma(\mathbf{w}_l) - \sum_{i=1}^r d(\mathbf{w}_l, \Gamma(h_i)) s_i})}.$$

□

Remark 6. In the p -adic case, $K = \mathbb{Q}_p$, Theorem 1 is a generalization of Theorem 4.2 in [8] and Theorem 4.3 in [6].

5. LOCAL ZETA FUNCTION FOR RATIONAL FUNCTIONS

From now on, we fix two non-constant polynomials $f(\mathbf{x})$, $g(\mathbf{x})$ in n variables, $n \geq 2$, with coefficients in $\mathcal{O}_K[x_1, \dots, x_n] \setminus \pi \mathcal{O}_K[x_1, \dots, x_n]$ and set $D_K := \{\mathbf{x} \in K^n; f(\mathbf{x}) = 0\} \cup \{\mathbf{x} \in K^n; g(\mathbf{x}) = 0\}$, and

$$\frac{f}{g} : K^n \setminus D_K \rightarrow K.$$

Furthermore, we define the *Newton polyhedron* $\Gamma\left(\frac{f}{g}\right)$ of $\frac{f}{g}$ to be $\Gamma(fg)$, and assume that the mapping $(f, g) : K^n \rightarrow K^2$ is non-degenerate over \mathbb{F}_q with respect to $\Gamma\left(\frac{f}{g}\right)$ as before. In this case we will say that $\frac{f}{g}$ is *non-degenerate over \mathbb{F}_q with respect to $\Gamma\left(\frac{f}{g}\right)$* . We fix a simplicial polyhedral subdivision $\mathcal{F}\left(\frac{f}{g}\right)$ of \mathbb{R}_+^n subordinate to $\Gamma\left(\frac{f}{g}\right)$. For $\Delta \in \mathcal{F}\left(\frac{f}{g}\right) \cup \{\mathbf{0}\}$, we put

$$\begin{aligned} N_{\Delta, \{f\}} &:= \operatorname{Card} \left\{ \bar{\mathbf{a}} \in (\mathbb{F}_q^\times)^n; \bar{f}_{b(\Delta)}(\bar{\mathbf{a}}) = 0 \text{ and } \bar{g}_{b(\Delta)}(\bar{\mathbf{a}}) \neq 0 \right\}, \\ N_{\Delta, \{g\}} &:= \operatorname{Card} \left\{ \bar{\mathbf{a}} \in (\mathbb{F}_q^\times)^n; \bar{f}_{b(\Delta)}(\bar{\mathbf{a}}) \neq 0 \text{ and } \bar{g}_{b(\Delta)}(\bar{\mathbf{a}}) = 0 \right\}, \\ N_{\Delta, \{f, g\}} &:= \operatorname{Card} \left\{ \bar{\mathbf{a}} \in (\mathbb{F}_q^\times)^n; \bar{f}_{b(\Delta)}(\bar{\mathbf{a}}) = 0 \text{ and } \bar{g}_{b(\Delta)}(\bar{\mathbf{a}}) = 0 \right\}, \end{aligned}$$

with the convention that if $b(\Delta) = b(\mathbf{0}) = \mathbf{0}$, then $f_{b(\Delta)} = f$ and $g_{b(\Delta)} = g$. We also define $\mathfrak{D}\left(\frac{f}{g}\right) = \mathfrak{D}(f, g)$, which is the set of primitive vectors in $\mathbb{N}^n \setminus \{\mathbf{0}\}$ perpendicular to the facets of $\Gamma\left(\frac{f}{g}\right)$. We set

$$\begin{aligned} T_+ &:= \left\{ \mathbf{w} \in \mathfrak{D}\left(\frac{f}{g}\right); d(\mathbf{w}, \Gamma(g)) - d(\mathbf{w}, \Gamma(f)) > 0 \right\}, \\ T_- &:= \left\{ \mathbf{w} \in \mathfrak{D}\left(\frac{f}{g}\right); d(\mathbf{w}, \Gamma(g)) - d(\mathbf{w}, \Gamma(f)) < 0 \right\}, \\ \alpha &:= \alpha\left(\frac{f}{g}\right) = \begin{cases} \min_{\mathbf{w} \in T_+} \left\{ \frac{\sigma(\mathbf{w})}{d(\mathbf{w}, \Gamma(g)) - d(\mathbf{w}, \Gamma(f))} \right\} & \text{if } T_+ \neq \emptyset \\ +\infty & \text{if } T_+ = \emptyset, \end{cases} \\ \beta &:= \beta\left(\frac{f}{g}\right) = \begin{cases} \max_{\mathbf{w} \in T_-} \left\{ \frac{\sigma(\mathbf{w})}{d(\mathbf{w}, \Gamma(g)) - d(\mathbf{w}, \Gamma(f))} \right\} & \text{if } T_- \neq \emptyset \\ -\infty & \text{if } T_- = \emptyset, \end{cases} \end{aligned}$$

and

$$\tilde{\alpha} := \tilde{\alpha}\left(\frac{f}{g}\right) = \min\{1, \alpha\}, \quad \tilde{\beta} := \tilde{\beta}\left(\frac{f}{g}\right) = \max\{-1, \beta\}.$$

Notice that $\alpha > 0$ and $\beta < 0$.

We define the local zeta function attached to $\frac{f}{g}$ as

$$Z\left(s, \frac{f}{g}\right) = Z(s, -s, f, g), \quad s \in \mathbb{C},$$

where $Z(s_1, s_2, f, g)$ denotes the meromorphic continuation of the local zeta function attached to the polynomial mapping (f, g) , see Theorem 1.

Theorem 2. *Assume that $\frac{f}{g}$ is non-degenerate over \mathbb{F}_q with respect to $\Gamma\left(\frac{f}{g}\right)$, with $n \geq 2$ as before. We fix a simplicial polyhedral subdivision $\mathcal{F}\left(\frac{f}{g}\right)$ of \mathbb{R}_+^n subordinate to $\Gamma\left(\frac{f}{g}\right)$. Then the following assertions hold:*

(i) *$Z\left(s, \frac{f}{g}\right)$ has a meromorphic continuation to the whole complex plane as a rational function of q^{-s} and the following explicit formula holds:*

$$Z\left(s, \frac{f}{g}\right) = \sum_{\Delta \in \mathcal{F}\left(\frac{f}{g}\right) \cup \{\mathbf{0}\}} L_{\Delta}\left(s, \frac{f}{g}\right) S_{\Delta}(s),$$

where for $\Delta \in \mathcal{F}\left(\frac{f}{g}\right) \cup \{\mathbf{0}\}$,

$$L_{\Delta}\left(s, \frac{f}{g}\right) = q^{-n} \left[(q-1)^n - N_{\Delta, \{f\}} \frac{1-q^{-s}}{1-q^{-1-s}} - N_{\Delta, \{g\}} \frac{1-q^s}{1-q^{-1+s}} - N_{\Delta, \{f, g\}} \frac{(1-q^{-s})(1-q^s)}{q(1-q^{-1-s})(1-q^{-1+s})} \right]$$

and

$$S_{\Delta}(s) = \frac{\sum_{\mathbf{t}} q^{-\sigma(\mathbf{t}) - (d(\mathbf{t}, \Gamma(f)) - d(\mathbf{t}, \Gamma(g)))s}}{\prod_{i=1}^l (1 - q^{-\sigma(\mathbf{w}_i) - (d(\mathbf{w}_i, \Gamma(f)) - d(\mathbf{w}_i, \Gamma(g)))s})},$$

for $\Delta \in \mathcal{F}\left(\frac{f}{g}\right)$ a cone strictly positively generated by linearly independent vectors $\mathbf{w}_1, \dots, \mathbf{w}_l \in \mathfrak{D}\left(\frac{f}{g}\right)$, and where \mathbf{t} runs through the elements of the set

$$\mathbb{Z}^n \cap \left\{ \sum_{i=1}^l \lambda_i \mathbf{w}_i; \quad 0 < \lambda_i \leq 1 \text{ for } i = 1, \dots, l \right\}.$$

By convention $S_{\mathbf{0}}(s) := 1$.

(ii) *$Z\left(s, \frac{f}{g}\right)$ is a holomorphic function on $\tilde{\beta} < \operatorname{Re}(s) < \tilde{\alpha}$, and on this band it verifies that*

$$(5.1) \quad Z\left(s, \frac{f}{g}\right) = \int_{\mathcal{O}_K^n \setminus D_K} \left| \frac{f(x)}{g(x)} \right|^s |dx|;$$

(iii) the poles of the meromorphic continuation of $Z\left(s, \frac{f}{g}\right)$ belong to the set

$$\bigcup_{k \in \mathbb{Z}} \left\{ 1 + \frac{2\pi\sqrt{-1}k}{\ln q} \right\} \cup \bigcup_{k \in \mathbb{Z}} \left\{ -1 + \frac{2\pi\sqrt{-1}k}{\ln q} \right\} \cup \bigcup_{\mathbf{w} \in \mathfrak{D}\left(\frac{f}{g}\right)} \bigcup_{k \in \mathbb{Z}} \left\{ \frac{\sigma(\mathbf{w})}{d(\mathbf{w}, \Gamma(g)) - d(\mathbf{w}, \Gamma(f))} + \frac{2\pi\sqrt{-1}k}{\{d(\mathbf{w}, \Gamma(g)) - d(\mathbf{w}, \Gamma(f))\} \ln q} \right\}.$$

Proof. (i) The explicit formula for $Z(s, \frac{f}{g})$ follows from Theorem 1 as follows: we take $r = 2$, $s_1 = s$, $s_2 = -s$, $h_1 = f_{b(\Delta)}$ and $h_2 = g_{b(\Delta)}$ for $\Delta \in \mathcal{F}\left(\frac{f}{g}\right) \cup \{\mathbf{0}\}$, with the convention that if $b(\Delta) = b(\mathbf{0}) = \mathbf{0}$, then $h_1 = f$ and $h_2 = g$. Now

$$\overline{V}_\Delta = \left\{ \overline{\mathbf{z}} \in (\mathbb{F}_q^\times)^n; \overline{f}_{b(\Delta)}(\overline{\mathbf{z}}) = \overline{g}_{b(\Delta)}(\overline{\mathbf{z}}) = 0 \right\} \text{ for } \Delta \in \mathcal{F}\left(\frac{f}{g}\right) \cup \{\mathbf{0}\},$$

and thus $\text{Card}(\overline{V}_\Delta) = N_{\Delta, \{f, g\}}$. Now, with $I = \{1, 2\}$, by using (2.6), we have

$$(5.2) \quad J(s, -s, \overline{V}_\Delta) = \frac{q^{-n} (1 - q^{-1})^2 N_{\Delta, \{f, g\}}}{(1 - q^{-1-s})(1 - q^{-1+s})}.$$

We now consider the case $I \neq \emptyset$, $I \subsetneq \{1, 2\}$, thus there are two cases: $I = \{1\}$ or $I = \{2\}$. Note that

$$\overline{V}_{\Delta, \{1\}} = \left\{ \overline{\mathbf{z}} \in (\mathbb{F}_q^\times)^n; \overline{f}_{b(\Delta)}(\overline{\mathbf{z}}) = 0 \text{ and } \overline{g}_{b(\Delta)}(\overline{\mathbf{z}}) \neq 0 \right\} \text{ for } \Delta \in \mathcal{F}\left(\frac{f}{g}\right) \cup \{\mathbf{0}\},$$

and that $\text{Card}(\overline{V}_{\Delta, \{1\}}) = N_{\Delta, \{f\}}$, with the convention that

$$\overline{V}_{\mathbf{0}, \{1\}} = \left\{ \overline{\mathbf{z}} \in (\mathbb{F}_q^\times)^n; \overline{f}(\overline{\mathbf{z}}) = 0 \text{ and } \overline{g}(\overline{\mathbf{z}}) \neq 0 \right\}.$$

In this case, by using (2.5),

$$(5.3) \quad J(s, -s, \overline{V}_{\Delta, \{1\}}) = \frac{q^{-n-s} (1 - q^{-1}) N_{\Delta, \{f\}}}{1 - q^{-1-s}}.$$

Analogously,

$$(5.4) \quad J(s, -s, \overline{V}_{\Delta, \{2\}}) = \frac{q^{-n+s} (1 - q^{-1}) N_{\Delta, \{g\}}}{1 - q^{-1+s}}.$$

We now consider the case $I = \emptyset$, then

$$\overline{V}_{\Delta, \emptyset} = \left\{ \overline{\mathbf{z}} \in (\mathbb{F}_q^\times)^n; \overline{f}_{b(\Delta)}(\overline{\mathbf{z}}) \neq 0 \text{ and } \overline{g}_{b(\Delta)}(\overline{\mathbf{z}}) \neq 0 \right\} \text{ for } \Delta \in \mathcal{F}\left(\frac{f}{g}\right) \cup \{\mathbf{0}\},$$

with the convention that

$$\overline{V}_{\mathbf{0}, \emptyset} = \left\{ \overline{\mathbf{z}} \in (\mathbb{F}_q^\times)^n; \overline{f}(\overline{\mathbf{z}}) \neq 0 \text{ and } \overline{g}(\overline{\mathbf{z}}) \neq 0 \right\}.$$

Notice that $\text{Card}(\overline{V}_{\Delta, \emptyset}) = (q - 1)^n - N_{\Delta, \{f\}} - N_{\Delta, \{g\}} - N_{\Delta, \{f, g\}}$. Then, by using (2.4),

$$(5.5) \quad J(s, -s, \overline{V}_{\Delta, \emptyset}) = q^{-n} \text{Card}(\overline{V}_{\Delta, \emptyset}).$$

Then from Theorem 1 and (5.2)-(5.5), we get

$$L_{\Delta}(s, \frac{f}{g}) = \frac{q^{-n} (1 - q^{-1})^2 N_{\Delta, \{f, g\}}}{(1 - q^{-1-s})(1 - q^{-1+s})} + \frac{q^{-n-s} (1 - q^{-1}) N_{\Delta, \{f\}}}{1 - q^{-1-s}} + \frac{q^{-n+s} (1 - q^{-1}) N_{\Delta, \{g\}}}{1 - q^{-1+s}} + q^{-n} \{(q-1)^n - N_{\Delta, \{f\}} - N_{\Delta, \{g\}} - N_{\Delta, \{f, g\}}\}.$$

The announced formula for $L_{\Delta}(s, \frac{f}{g})$ is obtained from the above formula after some simple algebraic manipulations.

(ii) Notice that for $\mathbf{w} \in \mathfrak{D}(\frac{f}{g})$, $\frac{1}{1 - q^{-\sigma(\mathbf{w}) - (d(\mathbf{w}, \Gamma(f)) - d(\mathbf{w}, \Gamma(g)))s}}$ is holomorphic on $\sigma(\mathbf{w}) + (d(\mathbf{w}, \Gamma(f)) - d(\mathbf{w}, \Gamma(g))) \operatorname{Re}(s) > 0$, and that $\frac{1}{1 - q^{-1-s}}$ is holomorphic on $\operatorname{Re}(s) > -1$, and $\frac{1}{1 - q^{-1+s}}$ is holomorphic on $\operatorname{Re}(s) < 1$, then, from the explicit formula for $Z(s, \frac{f}{g})$ given in (i) follows that it is holomorphic on the band $\tilde{\beta} < \operatorname{Re}(s) < \tilde{\alpha}$. Now, since $Z(s, \frac{f}{g}) = Z(s, -s, f, g)$, $Z(s, \frac{f}{g})$ is given by integral (5.1) because $Z(s_1, s_2, f, g)$ agrees with an integral on its natural domain.

(iii) It is a direct consequence of the explicit formula. \square

6. THE LARGEST AND SMALLEST REAL PART OF THE POLES OF $Z(s, \frac{f}{g})$ (DIFFERENT FROM -1 AND 1 , RESPECTIVELY)

In this section we use all the notation introduced in Section 5. We work with a fix simplicial polyhedral subdivision $\mathcal{F}(\frac{f}{g})$ of \mathbb{R}_+^n subordinate to $\Gamma(\frac{f}{g})$. We recall that in the case $T_- \neq \emptyset$,

$$\beta = \max_{\mathbf{w} \in T_-} \left\{ \frac{\sigma(\mathbf{w})}{d(\mathbf{w}, \Gamma(g)) - d(\mathbf{w}, \Gamma(f))} \right\}$$

is the largest possible ‘non-trivial’ negative real part of the poles of $Z(s, \frac{f}{g})$. We set

$$\mathcal{P}(\beta) := \left\{ \mathbf{w} \in T_-; \frac{\sigma(\mathbf{w})}{d(\mathbf{w}, \Gamma(g)) - d(\mathbf{w}, \Gamma(f))} = \beta \right\},$$

and for $m \in \mathbb{N}$ with $1 \leq m \leq n$,

$$\mathcal{M}_m(\beta) := \left\{ \Delta \in \mathcal{F}(\frac{f}{g}); \Delta \text{ has exactly } m \text{ generators belonging to } \mathcal{P}(\beta) \right\},$$

and $\rho := \max \{m; \mathcal{M}_m(\beta) \neq \emptyset\}$.

Theorem 3. *Suppose that $\frac{f}{g}$ is non-degenerated over \mathbb{F}_q with respect to $\Gamma(\frac{f}{g})$ and that $T_- \neq \emptyset$. If $\beta > -1$, then β is a pole of $Z(s, \frac{f}{g})$ of multiplicity ρ .*

Proof. In order to prove that β is a pole of $Z(s, \frac{f}{g})$ of order ρ , it is sufficient to show that

$$\lim_{s \rightarrow \beta} (1 - q^{\beta-s})^{\rho} Z\left(s, \frac{f}{g}\right) > 0.$$

This assertion follows from the explicit formula for $Z(s, \frac{f}{g})$ given in Theorem 2, by the following claim:

Claim. $\operatorname{Res}(\Delta, \beta) := \lim_{s \rightarrow \beta} (1 - q^{s-\beta})^{\rho} L_{\Delta}(s, \frac{f}{g}) S_{\Delta}(s) \geq 0$ for every cone $\Delta \in \mathcal{F}(\frac{f}{g})$. Furthermore, there exists a cone $\Delta_0 \in \mathcal{M}_{\rho}(\beta)$ such that $\operatorname{Res}(\Delta_0, \beta) > 0$.

We show that for at least one cone Δ_0 in $\mathcal{M}_\rho(\beta)$, $\text{Res}(\Delta_0, \beta) > 0$, because for any cone $\Delta \notin \mathcal{M}_\rho(\beta)$, $\text{Res}(\Delta, \beta) = 0$. This last assertion can be verified by using the argument that we give for the cones in $\mathcal{M}_\rho(\beta)$. We first note that there exists at least one cone Δ_0 in $\mathcal{M}_\rho(\beta)$. Let $\mathbf{w}_1, \dots, \mathbf{w}_\rho, \mathbf{w}_{\rho+1}, \dots, \mathbf{w}_l$ its generators with $\mathbf{w}_i \in \mathcal{P}(\beta) \Leftrightarrow 1 \leq i \leq \rho$.

On the other hand,

$$(6.1) \quad \lim_{s \rightarrow \beta} L_\Delta \left(s, \frac{f}{g} \right) > 0$$

for all cones $\Delta \in \mathcal{F}(\frac{f}{g}) \cup \{\mathbf{0}\}$. Inequality (6.1) follows from

$$L_\Delta \left(\beta, \frac{f}{g} \right) > q^{-n}((q-1)^n - N_{\Delta, \{f\}} - N_{\Delta, \{g\}} - N_{\Delta, \{f, g\}}) \geq 0$$

for all cones $\Delta \in \mathcal{F}(\frac{f}{g}) \cup \{\mathbf{0}\}$. We prove this last inequality in the case $N_{\Delta, \{f\}} > 0$, $N_{\Delta, \{g\}} > 0$, $N_{\Delta, \{f, g\}} > 0$ since the other cases are treated in similar form. In this case, the inequality follows from the formula for $L_\Delta(\beta, \frac{f}{g})$ given in Theorem 2, by using that

$$\begin{aligned} N_{\Delta, \{f\}} \frac{1 - q^{-\beta}}{1 - q^{-1-\beta}} &< N_{\Delta, \{f\}}, \quad N_{\Delta, \{g\}} \frac{1 - q^\beta}{1 - q^{-1+\beta}} < N_{\Delta, \{g\}}, \\ N_{\Delta, \{f, g\}} \frac{(1 - q^{-\beta})(1 - q^\beta)}{q(1 - q^{-1-\beta})(1 - q^{-1+\beta})} &< N_{\Delta, \{f, g\}} \quad \text{when } \beta > -1. \end{aligned}$$

We also notice that

$$\lim_{s \rightarrow \beta} \sum_{\mathbf{t}} q^{-\sigma(\mathbf{t}) - (d(\mathbf{t}, \Gamma(f)) - d(\mathbf{t}, \Gamma(g)))s} > 0.$$

Hence in order to show that $\text{Res}(\Delta_0, \beta) > 0$, it is sufficient to show that

$$\lim_{s \rightarrow \beta} \frac{(1 - q^{s-\beta})^\rho}{\prod_{i=1}^l (1 - q^{-\sigma(\mathbf{w}_i) - (d(\mathbf{w}_i, \Gamma(f)) - d(\mathbf{w}_i, \Gamma(g)))s})} > 0.$$

Now, notice that there are positive integer constants c_i such that

$$\begin{aligned} \prod_{i=1}^\rho (1 - q^{-\sigma(\mathbf{w}_i) - (d(\mathbf{w}_i, \Gamma(f)) - d(\mathbf{w}_i, \Gamma(g)))s}) &= \prod_{i=1}^\rho (1 - q^{(s-\beta)c_i}) \\ &= (1 - q^{s-\beta})^\rho \prod_{i=1}^\rho \prod_{\zeta^{c_i}=1, \zeta \neq 1} (1 - \zeta q^{s-\beta}). \end{aligned}$$

In addition, for $i = \rho + 1, \dots, l$,

$$1 - q^{-\sigma(\mathbf{w}_i) - (d(\mathbf{w}_i, \Gamma(f)) - d(\mathbf{w}_i, \Gamma(g)))\beta} > 0$$

because $-\sigma(\mathbf{w}_i) - (d(\mathbf{w}_i, \Gamma(f)) - d(\mathbf{w}_i, \Gamma(g)))\beta \leq 0$ for any $\mathbf{w}_i \in T_+ \cup T_-$ with $i = \rho + 1, \dots, l$. From these observations, we have

$$\begin{aligned} & \lim_{s \rightarrow \beta} \frac{(1 - q^{s-\beta})^\rho}{\prod_{i=1}^l (1 - q^{-\sigma(\mathbf{w}_i) - (d(\mathbf{w}_i, \Gamma(f)) - d(\mathbf{w}_i, \Gamma(g)))s})} = \\ & \lim_{s \rightarrow \beta} \frac{(1 - q^{s-\beta})^\rho}{(1 - q^{s-\beta})^\rho \prod_{i=1}^\rho (1 - \zeta q^{s-\beta})} \times \prod_{\zeta^{c_i}=1, \zeta \neq 1} (1 - \zeta q^{s-\beta})^\times \\ & \lim_{s \rightarrow \beta} \frac{1}{\prod_{i=\rho+1}^l (1 - q^{-\sigma(\mathbf{w}_i) - (d(\mathbf{w}_i, \Gamma(f)) - d(\mathbf{w}_i, \Gamma(g)))s})} > 0. \end{aligned}$$

□

In the case $T_+ \neq \emptyset$,

$$\alpha = \min_{\mathbf{w} \in T_+} \left\{ \frac{\sigma(\mathbf{w})}{d(\mathbf{w}, \Gamma(g)) - d(\mathbf{w}, \Gamma(f))} \right\}.$$

is the smallest possible ‘non-trivial’ positive real part of the poles of $Z(s, \frac{f}{g})$. We set

$$\mathcal{P}(\alpha) := \left\{ \mathbf{w} \in T_+; \frac{\sigma(\mathbf{w})}{d(\mathbf{w}, \Gamma(g)) - d(\mathbf{w}, \Gamma(f))} = \alpha \right\},$$

and for $m \in \mathbb{N}$ with $1 \leq m \leq n$,

$$\mathcal{M}_m(\alpha) := \left\{ \Delta \in \mathcal{F}\left(\frac{f}{g}\right); \Delta \text{ has exactly } m \text{ generators belonging to } \mathcal{P}(\alpha) \right\},$$

and $\kappa := \max \{m; \mathcal{M}_m(\alpha) \neq \emptyset\}$

The proof of the following result is similar to the proof of Theorem 3.

Theorem 4. *Suppose that $\frac{f}{g}$ is non-degenerated over \mathbb{F}_q with respect to $\Gamma(\frac{f}{g})$ and that $T_+ \neq \emptyset$. If $\alpha < 1$, then α is a pole of $Z(s, \frac{f}{g})$ of multiplicity κ .*

Example 4. *We compute the local zeta function for the rational function given in Example 1. With the notation of Theorem 2, one verifies that*

Cone	L_Δ	S_Δ
$\{\mathbf{0}\}$	$q^{-2}((q-1)^2 - (q-1)\frac{1-q^{-s}}{1-q^{-1-s}})$	1
Δ_1	$q^{-2}(q-1)^2$	$\frac{q^{-1+2s}}{1-q^{-1+2s}}$
Δ_2	$q^{-2}(q-1)^2$	$\frac{q^{-2+2s} + q^{-4+4s}}{(1-q^{-1+2s})(1-q^{-3+2s})}$
Δ_3	$q^{-2}((q-1)^2 - (q-1)\frac{1-q^{-s}}{1-q^{-1-s}})$	$\frac{q^{-3+2s}}{1-q^{-3+2s}}$
Δ_4	$q^{-2}(q-1)^2$	$\frac{q^{-4+3s}}{(1-q^{-3+2s})(1-q^{-1+s})}$
Δ_5	$q^{-2}(q-1)^2$	$\frac{q^{-1+s}}{(1-q^{-1+s})}$

Therefore

$$Z(s, \frac{f}{g}) = \frac{\frac{(q-1)}{q^2} L(q^{-s})}{(1 - q^{s-1})(1 - q^{-1-s})(1 - q^{2s-1})(1 - q^{2s-3})},$$

where

$$\begin{aligned} L(q^{-s}) &= q - q^{-1} - 2 - q^{2s-4} + q^{s-3} - q^{s-2} + q^{2s-2} + q^{3s-3} \\ &\quad + 2q^{2s-1} - q^{3s-2} - q^{3s-1} + q^{-s-1}. \end{aligned}$$

Furthermore, $Z(s, \frac{f}{g})$ has poles with real parts belonging to $\{-1, 1/2, 1, 3/2\}$.

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